SOLUTION SET

Chapter 3

PARTICLE NATURE OF LIGHT – DISCRETE ENERGY LEVELS

“LASER FUNDAMENTALS”

Second Edition

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1. Using the Bohr theory, carry out the derivation of the electron orbit radius $r$ (eqn. 3.4) and the orbital energy $E_n$ (eqn. 3.8) for the hydrogen atom.

From (3.2) \[ \mathbf{V} = \frac{n^2 \hbar^2}{m_e r} \Rightarrow \text{plug into (3.3)} \]

\[ \frac{e^2}{4 \pi \varepsilon_0 r^2} = \frac{m_e \left( \frac{n^2 \hbar^2}{m_e r^2} \right)}{r} = \frac{n^2 \hbar^2}{m_e r^3} \]

\[ \Rightarrow r = \frac{n^2 \hbar^2}{m_e e^2} = \frac{n^2 \left( \frac{2 \pi \hbar}{e} \right)^2 \varepsilon_0}{m_e \pi e^2} = \frac{e_0 \hbar^2}{n^2 m_e e^2} \]

From (3.6)

\[ E = \frac{1}{2} m_e \mathbf{V}^2 - \frac{e^2}{4 \pi \varepsilon_0 r} \]

but from (3.3), \[ \frac{1}{2} m_e \mathbf{V}^2 = \frac{1}{2} \frac{e^2 r}{4 \pi \varepsilon_0 r^2} = \frac{e^2}{8 \pi \varepsilon_0 r} \]

Hence \[ E = - \frac{e^2}{8 \pi \varepsilon_0 r} \]

Using $r$ from above:

\[ E = - \frac{e^2}{8 \pi \varepsilon_0 r} = - \frac{e^2}{8 \pi \varepsilon_0 \left( \frac{e_0 \hbar^2}{n^2 m_e e^2} \right)} \]

\[ = - \frac{m_e e^4}{8 \varepsilon_0^2 h^2 n^2} = - \frac{E_0}{n^2} \]

where \[ E_0 = \frac{m_e e^4}{8 \varepsilon_0^2 h^2} \]
2. Compute the wavelengths for the following transitions in the hydrogen atom using the Bohr theory:

(a) \( n = 2 \) to \( n = 1 \);
(b) \( n = 3 \) to \( n = 1 \);
(c) \( n = 4 \) to \( n = 2 \);
(d) \( n = 5 \) to \( n = 3 \);
(e) \( n = 6 \) to \( n = 4 \).

From (3.11) \[ \frac{1}{\lambda} = R_H \left( \frac{1}{n_i^2} - \frac{1}{n_f^2} \right) = \left( 1.0967758 \times 10^{-7} \right) \left( \frac{1}{n_i^2} - \frac{1}{n_f^2} \right) \]

(a) \( n_i = 2 \) to \( n_f = 1 \) \[ \frac{1}{\lambda_{21}} = \left( \frac{1}{2^2} - \frac{1}{1^2} \right) R_H \]
\[ \lambda_{21} = 1.2157 \times 10^{-7} \text{ m} = 121.57 \text{ nm} \]

(b) \( n_i = 3 \) to \( n_f = 1 \) \[ \frac{1}{\lambda_{31}} = \left( \frac{1}{3^2} - \frac{1}{1^2} \right) R_H \]
\[ \lambda_{31} = 1.0257 \times 10^{-7} \text{ m} = 102.57 \text{ nm} \]

(c) \( n_i = 4 \) to \( n_f = 2 \) \[ \frac{1}{\lambda_{42}} = \left( \frac{1}{4^2} - \frac{1}{2^2} \right) R_H \]
\[ \lambda_{42} = 4.8627 \times 10^{-7} \text{ m} = 486.27 \text{ nm} \]

(d) \( n_i = 5 \) to \( n_f = 3 \) \[ \frac{1}{\lambda_{53}} = \left( \frac{1}{5^2} - \frac{1}{3^2} \right) R_H \]
\[ \lambda_{53} = 1.2822 \times 10^{-6} \text{ m} = 1.2822 \text{ \mu m} \]

(e) \( n_i = 5 \) to \( n_f = 4 \) \[ \frac{1}{\lambda_{54}} = \left( \frac{1}{4^2} - \frac{1}{5^2} \right) R_H \]
\[ \lambda_{54} = 4.0523 \times 10^{-6} \text{ m} = 4.0523 \text{ \mu m} \]
3. Compute the ionization potential for five-times ionized carbon (C\(^{5+}\)) and also compute the wavelength of the transition from \(n = 3\) to \(n = 2\) for that species.

(a) From (3.13) \[ E_n = -\frac{Z^2 E_0}{n^2} \]

Ionization potential from ground state of C\(^{5+}\) is:

\[ E_1 = -\frac{Z^2 E_0}{1^2} = -Z^2 E_0 \]

but \(E_0 = 13.595\ eV\) and \(Z = 6\)

hence \(E_1 = 6^2 (13.595\ eV) = 489.42\ eV\)

or \(7.83 \times 10^{-17}\ J\)

(b) \(\lambda_{32}\) for H-like C\(^{5+}\) is from (3.14):

\[ \nu_{32} = \frac{\lambda}{\lambda_{32}} = \frac{Z^2 E_0}{\hbar} \left( \frac{1}{2^2} - \frac{1}{3^2} \right) \]

or \(\frac{1}{\lambda_{32}} = \frac{Z^2 E_0}{c \hbar} \left( \frac{1}{2^2} - \frac{1}{3^2} \right) \)

\[ \therefore \lambda_{32} = \frac{1}{R_H} \frac{1}{Z^2} \left( \frac{1}{\frac{1}{2^2} - \frac{1}{3^2}} \right) \]

\[ = 1.8235 \times 10^{-8}\ m = 18.235\ nm \]
4. In the isoelectronic scaling of hydrogen-like species, what is the lowest–atomic number element that would have a possible X-ray laser transition at a wavelength shorter than 4.4 nm? (Here, consider the transition from \( n = 3 \) to \( n = 2 \) as the laser transition.)

\[
\lambda_{32} = \frac{1}{z^2} \frac{1}{R_H} \left( \frac{1}{\frac{1}{2^2} - \frac{1}{3^2}} \right) = \frac{1}{z^2} \frac{1}{1.097 \times 10^7 \text{m}^{-1} \left( \frac{1}{\frac{1}{2^2} - \frac{1}{3^2}} \right)}
\]

\[
= \frac{1}{z^2} \left( 6.563 \times 10^{-7} \text{m} \right) = \frac{1}{z^2} \left( 656.3 \text{ nm} \right)
\]

Now \( \lambda_{32} < 4.4 \text{ nm} \)

\[
\frac{1}{z^2} \left( 656.3 \text{ nm} \right) < 4.4 \text{ nm}
\]

Hence \( z > \sqrt{\frac{656.3}{4.4}} = 12.21 \)

Since \( z \) is an integer, the next larger integer is 13

\( \therefore z = 13 \) which from the appendix on page 621 is Aluminum
5. Solve the Schrödinger equation for the energy eigenvalue of the ground state of the hydrogen atom, assuming a solution of the form \( \psi_1(r) = Ce^{-\alpha r} \) (eqn. 3.38).

Schrödinger Equation:

\[
\nabla^2 \psi + \frac{8\pi^2 \mu}{\hbar^2} (E - V) \psi = 0 \quad (3.29)
\]

for \( \psi = Ce^{-\alpha r} \) and \( V = \frac{-e^2}{4\pi \varepsilon_0 r} \)

\[
\nabla^2 (Ce^{-\alpha r}) + \frac{8\pi^2 \mu}{\hbar^2} (E_1 + \frac{e^2}{4\pi \varepsilon_0 r}) e^{-\alpha r} = 0
\]

Only \( r \) involved

\[
\nabla^2 (Ce^{-\alpha r}) = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} (Ce^{-\alpha r}) \right) = \frac{-\alpha Ce^{-\alpha r}}{r} \left[ -\alpha r + 2 \right]
\]

\[
(\alpha^2 + \frac{8\pi^2 \mu E_1}{\hbar^2}) e^{-\alpha r} + \left( \frac{c^2 \pi m e^2}{\varepsilon_0 \hbar^2} - 2\alpha c \right) e^{-\alpha r} = 0
\]

Each parenthesis must equal zero for all \( r \)

\[
\frac{c^2 \pi m e^2}{\varepsilon_0 \hbar^2} - 2\alpha c = 0 \Rightarrow \alpha = \frac{\pi m e^2}{\varepsilon_0 \hbar^2} = \frac{1}{a_H}
\]

and \( \alpha^2 + \frac{c^8 \pi^2 \mu E_1}{\hbar^2} = 0 \Rightarrow E_1 = \frac{-\alpha^2 \hbar^2}{8 \pi^2 \mu} = -\frac{m e^4}{8 \pi^2 \hbar^2} = E_1
\]

Normalize to find \( C \)

\[
\int |\psi|^2 dV = \int_0^\infty C^2 e^{-2\alpha r} 4\pi r^2 dr = \frac{C^2 2.4\pi}{(2\alpha)^3} = 1
\]

\[
\Rightarrow C = \frac{\alpha^{3/2}}{\sqrt{\pi}} = \frac{1}{a_H^{3/2} \sqrt{\pi}} \Rightarrow \psi_1 = \frac{1}{\sqrt{\pi} \ a_H^{3/2}} e^{-\alpha r/a_H}
\]
6. Write out the wave function for the hydrogen atom for \( n = 3, l = 2, \) and \( m = +1 \) using (3.65) and (3.66).

\[
\begin{align*}
    n &= 3, \quad l = 2, \quad m = +1 \\
    from (3.65) \quad \psi_{n,l,m} &= C \rho^l e^{-\frac{\rho^2}{2}} L_{n+l}^l (\rho) P_l^m (\cos \theta) e^{i m \phi} \\
    \rho &= \frac{2r}{n a_H} \\
    \psi_{3,2,1} &= C \rho^2 e^{-\rho^2/2} L_5^2 P_2^1 (\cos \theta) e^{i \phi} \quad \rho = \frac{2r}{3 a_H}
\end{align*}
\]

\[
from (3.66) \quad C = -\left\{ \left( \frac{2}{n a_H} \right)^3 \left( \frac{(n-l)!}{2n [(n+l)!]} \right)^{1/2} \right\} \left[ \frac{(2l+1)(l-1m)!}{4\pi (l+1m)!} \right]^{1/2}
\]

\[
= -\left\{ \left( \frac{2}{3 a_H} \right)^3 \frac{0!}{6 (5)!^3} \right\}^{1/2} \left[ \frac{5}{4\pi} \frac{1!}{3!} \right]^{1/2}
\]

\[
= -\left\{ \frac{8}{27 a_H^3} \frac{1}{6 \cdot 120^3} \right\}^{1/2} \left[ \frac{5}{24\pi} \right]^{1/2}
\]

\[
= -\left\{ \frac{1}{27 \cdot 6 \cdot 15 \cdot 120^2 a_H^3 \cdot \frac{5}{24\pi}} \right\}^{1/2} = \frac{-12960 a_H^3 a_H^{-3/2}}{a_H^{3/2}} = -\frac{4.35 \times 10^{-5}}{a_H^{3/2}}
\]

From (3.57)

determine \( L_5^2 \quad L_n = n! \quad \therefore L_5^2 = -120 \\
From (3.54) \quad P_2^1 (x) = 3x (1-x^2)^{1/2}
\]

\[
\therefore P_2^1 \cos \theta = 3 \cos \theta (1-\cos^2 \theta)^{1/2} = 3 \cos \theta \sin \theta
\]

\[
\therefore \psi_{3,2,1} = -\frac{4.35 \times 10^{-5}}{a_H^{3/2}} \rho^2 e^{-\rho^2/2} (-120) 3 \cos \theta \sin \theta e^{i \phi}
\]

\[
= \left( 0.97 \times 10^{-3} \right) \frac{r^2}{a_H^{11/2}} e^{-\frac{r^2}{2 a_H}} \cos \theta \sin \theta e^{i \phi}
\]
7. Obtain the possible term designations for a hydrogen atom in which the electron is in the 3d level.

3d level of hydrogen

\[ s = \frac{1}{2}, \quad l = 2 \quad \Rightarrow \quad J = \frac{5}{2}, \frac{3}{2} \]

\[ s = \frac{1}{2}, \quad l = 2 \]

\[ J = \frac{5}{2}, \frac{3}{2} \]

\[ 2 \text{D}\frac{5}{2}, \quad 2 \text{D}\frac{3}{2} \]
8. Give the electron configuration for a lithium atom in its ground state, and write out the term designation for that state.

\[
1s^2 \ 2s^1 \text{ Li ground state}
\]

for 2s

\[
\text{for } 2s
\]

\[
S = \frac{1}{2} \quad l = 0 \implies \text{S state}
\]

\[
2S+1 \quad L \implies 2 \quad S \frac{1}{2}
\]
9. List the electron configurations for the following elements in their ground states:
aluminum, fluorine, calcium, germanium, and krypton.

**Aluminum**
\[ 1s^2 2s^2 2p^6 3s^2 3p^1 \]

**Fluorine**
\[ 1s^2 2s^2 2p^5 \]

**Calcium**
\[ 1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 \]

**Germanium**
\[ 1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^2 \]

**Krypton**
\[ 1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 \]
10. Determine the number of states and write out the term designations (eqn. 3.76) for those states that result from an outer electron configuration of 2s4p.

\[
\begin{align*}
\text{2s} & \Rightarrow l=0 \quad s=\frac{1}{2} \\
\text{4p} & \Rightarrow l=1 \quad s=\frac{1}{2}
\end{align*}
\]

\[L = l_1 + l_2 = 0 + 1 = 1\]

\[S = s_1 + s_2 = \frac{1}{2} + \frac{1}{2} = 1\]

\[= s_1 - s_2 = \frac{1}{2} - \frac{1}{2} = 0\]

\[L=1 \quad S=1\]

\[J = L + S = 1 + 1 = 2 \Rightarrow \,^3P_2\]

\[1 + 0 = 1 \Rightarrow \,^3P_1\]

\[1 - 1 = 0 \Rightarrow \,^3P_0\]

\[L=1 \quad S=0\]

\[J = L + S = 1 + 0 = 1 \Rightarrow \,^1P_1\]
11. You are told that a short-wavelength laser is measured to have a wavelength of 30.38 nm and occurs on an H-like transition from $n = 4$ to $n = \frac{8}{2}$. In what element does the laser action occur? Justify your answer.

$$\nu = \frac{Z^2 E_0}{\hbar} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) = Z^2 \frac{3.28 \times 10^{15}}{30.38 \times 10^{-9}} \left( \frac{1}{2^2} - \frac{1}{4^2} \right)$$

$$= Z^2 \frac{6.15 \times 10^{14}}{6.15 \times 10^{14}}$$

$$Z^2 = \frac{9.87 \times 10^{15}}{6.15 \times 10^{14}} = 16.048$$

$$\therefore Z = 4.00 \Rightarrow [\text{Be}]$$
12. The element magnesium (Mg) is vaporized to produce a gas of magnesium atoms. An electrical current is passed through the gas in order to excite the atoms to levels above the ground state. Give the complete ground-state electron configuration. Make an energy-level diagram and provide the labels of the ground state and all of the excited energy levels associated with the same $n$ quantum number as that of the ground state. Allow only one of the outer electrons to be excited.
13. Obtain the spectroscopic designation and diagram the energy levels for the following three electronic configurations:

\[ 1s^2 2s^2, \quad 1s^2 2s 2p, \quad 1s^2 2s 3d. \]

\[ 1s^2 2s^2 \rightarrow S = \frac{1}{2} - \frac{1}{2}, \quad L = 0, 0 \]

\[ 1s^2 2s 2p \rightarrow S = \frac{1}{2}, \quad L = 0 \]

\[ S = \frac{1}{2} + \frac{1}{2} = 1, \quad L = 0 + 1 = 1 \]

\[ 1s^2 2s 3d \rightarrow S = \frac{1}{2}, \quad L = 0 \]

\[ S = \frac{1}{2} + \frac{1}{2} = 1, \quad L = 0 + 2 = 2 \]
14. Determine the number of energy levels and the term designations for those levels of an atom in a state in which the two outer electrons are 2s and 2p.

\[ 2s\,2p \]

\[
\begin{align*}
    s &= \frac{1}{2}, \quad l = 0 \\
    s &= \frac{1}{2}, \quad l = 1 \\
    s &= \frac{1}{2} + \frac{1}{2} = 1 \quad L = 0 + 1 = 1 \quad J = 2, 1, 0 \\
    s &= \frac{1}{2} - \frac{1}{2} = 0 \quad L = 0 + 1 = 1 \quad J = 1
\end{align*}
\]

\[ ^3P^0 \]

\[ ^1P^0 \]

41 states